CHAPTER 7:

SECOND ORDER DERIVATIVES

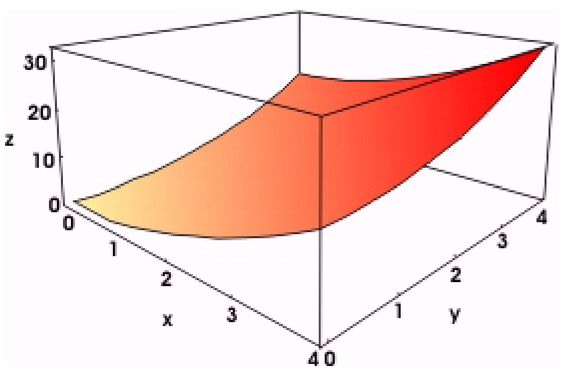
7.1 THE SECOND ORDER DERIVATE f_{xx} (x , y)

We have already established that $f_x(x, y)$ represents the slope in the x direction. If the derivative of $f_x(x, y)$ with respect to x is taken, the result is:

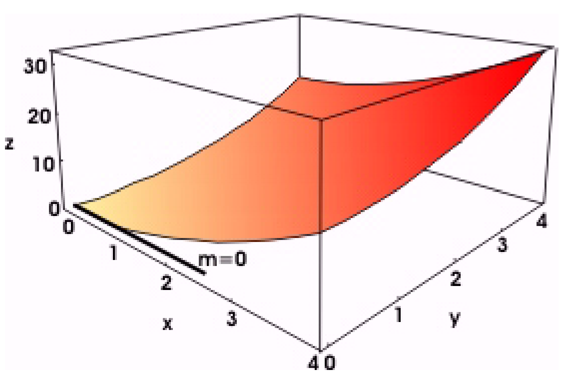
$$\frac{\partial f_x(x,y)}{\partial x} \approx \frac{\Delta f_x(x,y)}{\partial x} \approx \frac{\Delta m_x}{\Delta x}$$

Geometrically, this is the rate at which the slope in the x direction is changing as we move in the x direction.

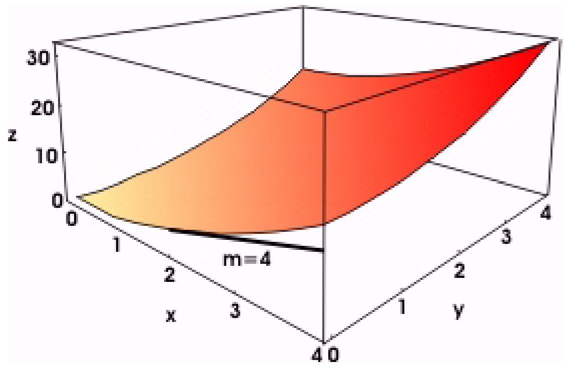
To visualize this geometrically, we can start by observing the graph of $f(x, y) = x^2 + y^2$ shown in the graph below.



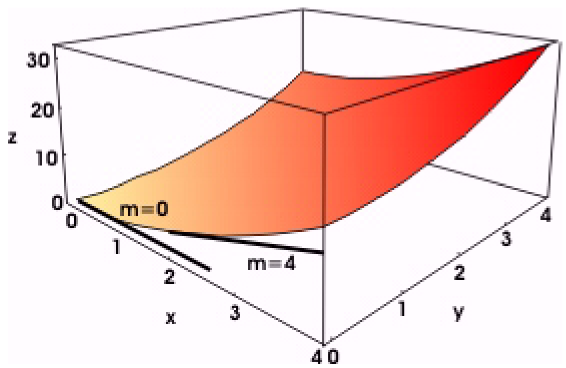
By taking the partial derivate with respect to x and evaluating it at the appropriate point, we can conclude that (0, 0, 0) the slope in the x direction is 0.



By taking the partial derivate with respect to x and evaluating it at the appropriate point, we can conclude that at (2, 0, 4), the slope in the x direction is 4.



Correspondingly, the change in $f_x(x, y)$ is 4 as we move 2 units in the *x* direction.



Hence, we can conclude that

$$\frac{\partial f_x(x,y)}{\partial x} \approx \frac{\Delta f_x(x,y)}{\partial x} \approx \frac{\Delta m_x}{\Delta x} = \frac{4}{2} = 2$$

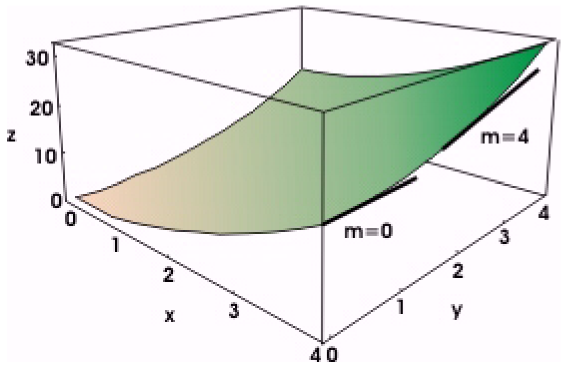
7.2 THE SECOND ORDER DERIVATE f_{yy} (x , y)

We have already established that $f_y(x, y)$ represents the slope in the *y* direction. If the derivative of $f_y(x, y)$ with respect to *y* is taken, the result is:

$$\frac{\partial f_y(x,y)}{\partial y} \approx \frac{\Delta f_y(x,y)}{\partial y} \approx \frac{\Delta m_y}{\Delta y}$$

Geometrically, this is the rate at which the slope in the *y* direction is changing as we move in the *y* direction.

To visualize this geometrically, we can follow the same procedure we followed with $f_{xx}(x, y)$ and the surface $f(x, y) = x^2 + y^2$. By taking the partial derivate with respect to *y* and evaluating it at the appropriate point, we can conclude that (4, 0, 16), the slope in the *y* direction is 0. By taking the partial derivate with respect to *y* and evaluating it at the appropriate point, we can conclude that at (4, 2, 20), the slope in the *y* direction is 4.



Correspondingly, the change in f_y (x, y) is 4 as we move 2 units in the *y* direction. Hence we can conclude that

$$\frac{\partial f_y(x,y)}{\partial y} \approx \frac{\Delta f_y(x,y)}{\partial y} \approx \frac{\Delta m_y}{\Delta y} = \frac{4}{2} = 2$$

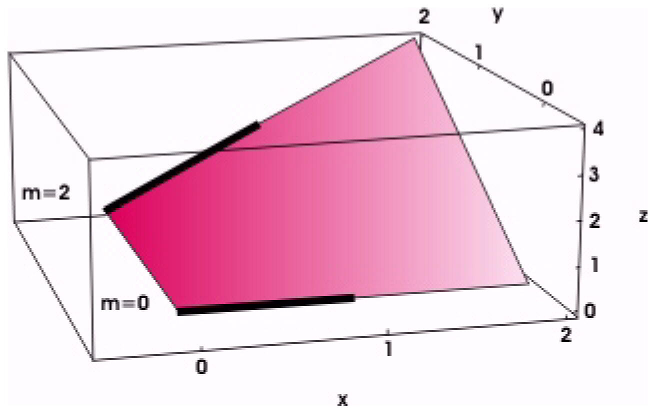
7.3 THE SECOND ORDER DERIVATE f_{xy} (x , y)

We have already established that f_x (x, y) represents the slope in the *x* direction. If the derivative of f_x (x, y) with respect to *x* is taken, the result is:

$$\frac{\partial f_x(x,y)}{\partial x} \approx \frac{\Delta f_x(x,y)}{\partial x} \approx \frac{\Delta m_x}{\Delta x}$$

Geometrically, this is the rate at which the slope in the *x* direction is changing as we move in the *x* direction.

We can follow the same procedure to approximate f_{xy} (x, y) that we followed in previous sections to determine f_{xy} $(0, 0)$ on the surface $f(x, y) = xy$. The slope of the tangent line in the *x* direction at the point (0, 0, 0) is equal to zero. The slope of the tangent line in the *x* direction at the point (0, 2, 0) on the surface $f(x, y) = xy$ is equal to two.



Correspondingly, we can conclude that

$$\frac{\partial f_x(x,y)}{\partial y} \approx \frac{\Delta f_x(x,y)}{\partial y} \approx \frac{\Delta m_x}{\Delta y} = \frac{2}{2} = 1$$